Supplementary materials

Appendix A

Table A1: The results of the in-sample estimation of EGARCH model and FFF-EGARCH model for Brent crude oil market

	EGARCH	FFF-EGARCH				
		k=1	k=2	k=3	k=4	k=5
δ	0.0835***(2.9235)	0.0836***(2.9308)	0.0845***(2.9694)	0.0848***(2.9911)	0.0846***(2.9833)	0.0872***(3.0741)
α_0	-0.0575***(-2.7179)	-0.0530***(-5.5648)	-0.0386***(-3.5497)	-0.0289***(-2.3869)	-0.0204***(-1.5152)	-0.0097***(-0.6128)
α_1	0.0959***(7.4906)	0.0963***(7.2803)	0.0997***(6.8262)	0.1012***(6.5899)	0.0049***(6.3864)	0.1073***(6.3187)
β_1	0.9894***(343.0098)	0.9862***(283.2876)	0.9749***(172.6856)	0.9676***(135.9058)	0.9615***(116.2504)	0.9517***(90.0232)
ξ	-0.0245***(-3.2877)	-0.0273***(-3.5249)	-0.0345***(-3.9474)	-0.0387***(-4.1511)	-0.0438***(-4.4019)	-0.0465***(-4.3596)
v	7.5899***(10.2267)	7.6477***(10.0412)	7.8150***(9.8281)	7.9008***(9.6993)	8.0013***(5.6094)	8.0791***(9.3998)
ϕ_1		0.0015(0.8408)	0.0031*(1.3603)	0.0041*(1.5709)	0.0049**(1.7133)	0.0063**(1.8687)
ψ_1		-0.0039**(-1.9210)	-0.0070***(-2.5998)	-0.0091***(-2.8875)	-0.0109***(-3.0920)	-0.0135***(-3.2228)
ϕ_2			-0.0011(-0.5099)	-0.0015(-0.6255)	-0.0020(-0.7474)	-0.0029(-0.9116)
ψ_2			-0.0099***(-3.1640)	-0.0128***(3.4006)	-0.0151***(-3.5508)	-0.0190***(-3.6508)
ϕ_3				0.0016(0.68371)	0.0021(0.7941)	0.0027(0.8761)
ψ_3				-0.0070**(-2.3061)	-0.0082***(-2.4193)	-0.0102***(-2.5454)
ϕ_4					0.0064**(2.0990)	-0.0102**(-2.2130)

	EGARCH	FFF-EGARCH				
		$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
ψ_4					0.0018(0.6747)	0.0022(0.7062)
ϕ_5						-0.0053*(-1.5534)
ψ_5						0.0046*(1.3378)
LL	-9873.9572	-9871.5938	-9864.286	-9860.5166	-9857.7246	**-9855.4819**
AIC	4.3371	4.3369	4.3346	4.3338	4.3335	**4.3333**
SC	**4.3441**	4.3462	4.3463	4.3478	4.3498	4.3520
$Q(10)$	9.870[0.4519]	9.973[0.4428]	8.731[0.5578]	8.837[0.5476]	8.356[0.5940]	8.084[0.6206]
$Q(20)$	20.978[0.3984]	21.564[0.3646]	19.507[0.4891]	19.406[0.4955]	19.022[0.5203]	18.464[0.5568]
$Q^2(10)$	33.083[0.0002]	27.789[0.0019]	22.996[0.0107]	22.365[0.0133]	22.606[0.0122]	17.661[0.0609]
$Q^2(20)$	39.497[0.0057]	33.528[0.0295]	28.600[0.0959]	27.720[0.1161]	28.097[0.1071]	23.233[0.2774]

Note: The *t*-statistics values and *p*-values are reported in the parentheses and square brackets, respectively. ***, ** and * denote the significance at the 1%, 5% and 10% levels, respectively. LL is the log-likelihood value. AIC means the Akaike Information Criterion, while SC means the Schwarz Criterion. *k* is a constant which controls how many cumulative Fourier frequencies are included in the approximation of FFF-GARCH-type models. $Q(10)$, $Q(20)$ and $Q^2(10)$, $Q^2(20)$ denote the Ljung and Box statistics of the standardized residuals and squared residuals for up to 10th and 20th order serial autocorrelation, respectively.

Table A2: The results of the in-sample estimation of GJR-GARCH model and FFF-GJR-GARCH model for Brent crude oil market

	GJR-GARCH	FFF-GJR-GARCH				
		k=1	k=2	k=3	k=4	k=5
δ	0.0868***(3.0218)	0.0866***(3.0299)	0.0866***(3.0354)	0.0862***(3.0229)	0.0851***(2.9832)	0.0872***(3.0614)
α_0	0.0453***(3.4920)	0.0866***(4.1580)	0.1467***(4.5593)	0.1583***(4.6698)	0.1693***(4.6983)	0.1937***(4.6609)
α_1	0.0254***(3.3693)	0.0247***(3.1135)	0.0218***(2.6017)	0.0198***(2.3680)	0.0189**(2.2563)	0.0197***(2.2838)
β_1	0.9512***(140.4541)	0.9403***(107.2678)	0.9267***(81.9492)	0.9246***(79.0231)	0.9224***(76.2131)	0.9163***(69.2084)
ξ	0.0280***(3.0827)	0.0342***(3.4420)	0.0429***(3.8439)	0.0458***(4.0379)	0.0476***(4.1037)	0.0488***(4.0373)
v	7.7405***(9.8662)	7.9409***(9.5037)	8.1445***(9.3328)	8.1979***(9.2940)	8.2422***(9.2334)	8.2763***(4.0373)
ϕ_1		0.0142*(1.5230)	0.0312**(2.1887)	0.0332**(2.2109)	0.0331**(2.1059)	0.0348**(2.0465)
ψ_1		-0.0322***(-3.0377)	-0.038***(-3.0688)	-0.0339***(-2.6946)	-0.0371***(-2.8460)	-0.0425***(-2.9382)
ϕ_2			-0.0052(-0.5015)	0.0006(0.0594)	-0.0043(-0.3517)	-0.0045(-0.3352)
ψ_2			-0.0465***(-3.2349)	-0.0455***(-3.1054)	-0.0530***(-3.2071)	-0.0608***(-3.2696)
ϕ_3				-0.0055(-0.5062)	-0.0112(-0.9518)	-0.0049(-0.3688)
ψ_3				-0.0234**(-1.9618)	-0.0246**(1.9033)	-0.0292**(-1.9214)
ϕ_4					0.0156*(1.3187)	0.0228*(1.6459)

	GJR-GARCH	FFF-GJR-GARCH				
		$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
ψ_4					0.0104(0.9021)	0.0060(0.471)
ϕ_5						-0.0206*(-1.5680)
ψ_5						0.0094(0.7296)
LL	-9869.19	-9861.9724	-9854.4242	-9852.0469	-9850.7064	**-9849.1172**
AIC	4.3350	4.3327	**4.3302**	4.3303	4.3304	4.3306
SC	4.3420	4.3420	**4.3419**	4.3441	4.3467	4.3492
$Q(10)$	10.124[0.4296]	9.801[0.4581]	8.715[0.5593]	8.963[0.5355]	8.770[0.5540]	8.519[0.5783]
$Q(20)$	20.851[0.4059]	21.176[0.3868]	19.265[0.5046]	19.539[0.4870]	19.294[0.5027]	18.921[0.5269]
$Q^2(10)$	23.023[0.0106]	14.618[0.1466]	11.198[0.3423]	12.950[0.2264]	13.426[0.2008]	10.738[0.3782]
$Q^2(20)$	31.107[0.0537]	21.591[0.3630]	17.893[0.5944]	19.047[0.5187]	19.710[0.4761]	16.695[0.6727]

Note: The *t*-statistics values and *p*-values are reported in the parentheses and square brackets, respectively. ***, ** and * denote the significance at the 1%, 5% and 10% levels, respectively. LL is the log-likelihood value. AIC means the Akaike Information Criterion, while SC means the Schwarz Criterion. k is a constant which controls how many cumulative Fourier frequencies are included in the approximation of FFF-GARCH-type models. $Q(10)$, $Q(20)$ and $Q^2(10)$, $Q^2(20)$ denote the Ljung and Box statistics of the standardized residuals and squared residuals for up to 10th and 20th order serial autocorrelation, respectively.

Table A3: The results of the in-sample estimation of GARCH model and FFF-GARCH model for WTI crude oil market

	GARCH	FFF-GARCH				
		k=1	k=2	k=3	k=4	k=5
δ	0.0886***(2.8361)	0.0881***(2.9935)	0.0889***(3.0658)	0.0896***(3.2573)	0.0884***(2.8491)	0.0888***(2.9163)
α_0	0.0839***(3.6500)	0.1170***(3.6513)	0.1544***(3.7756)	0.1647***(3.4038)	0.0171***(3.3360)	0.1868***(3.8262)
α_1	0.0498***(6.2742)	0.0507***(5.9027)	0.0512***(5.5179)	0.0516***(5.5735)	0.0516***(5.1925)	0.0515***(5.4410)
β_1	0.9361***(91.8584)	0.9092***(76.4577)	0.9220***(66.3310)	0.9197***(57.9501)	0.9185***(56.5282)	0.9161***(57.5225)
v	6.1504***(11.5687)	6.1889***(10.6272)	6.2377***(10.9316)	6.2584***(10.5419)	6.2589***(10.4309)	6.2896***(11.3473)
ϕ_1		0.0099(0.7866)	0.0223*(1.3179)	0.0196(0.9705)	0.0174(0.8311)	0.0181(0.9226)
ψ_1		-0.0391***(-2.6025)	-0.0401**(-2.3139)	-0.0390**(-2.0515)	-0.0404**(-2.1266)	-0.0465***(-2.6170)
ϕ_2			-0.0162(-1.1082)	-0.0155(1.0769)	-0.0192(-1.0505)	-0.0209(-1.2326)
ψ_2			-0.0401**(-2.2702)	-0.0378**(2.1086)	-0.0435**(-2.2434)	-0.0474***(-2.2330)
ϕ_3				0.0053(0.3797)	0.0012(0.0767)	0.0063(0.3896)
ψ_3				-0.0181*(-1.2217)	-0.0187*(-0.1915)	-0.0138(-0.8070)
ϕ_4					0.0167*(0.9165)	0.0283**(1.6056)
ψ_4					0.0076(0.4273)	0.0059(0.3840)
ϕ_5						-0.0264*(-1.6667)
ψ_5						-0.0080(-0.5282)

	GARCH	FFF-GARCH				
		$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
LL	-10020.7301	-10014.9518	-10010.0439	-10009.0405	-10008.1352	**-10006.1707**
AIC	4.4011	4.3994	**4.3981**	4.3986	4.3990	4.3991
SC	**4.4069**	4.4076	4.4086	4.4114	4.4142	4.4166
$Q(10)$	10.207[0.4225]	9.942[0.4455]	10.266[0.4174]	10.186[0.4243]	10.195[0.4235]	9.949[0.4450]
$Q(20)$	14.003[0.8303]	13.265[0.8657]	14.075[0.8266]	14.134[0.8236]	14.167[0.8219]	13.885[0.8362]
$Q^2(10)$	35.207[0.0002]	35.316[0.0001]	33.470[0.0002]	33.403[0.0002]	33.640[0.0002]	34.776[0.0001]
$Q^2(20)$	50.156[0.0002]	51.972[0.0001]	49.086[0.0002]	49.086[0.0003]	50.320[0.0001]	51.751[0.0001]

Note: The *t*-statistics values and *p*-values are reported in the parentheses and square brackets, respectively. ***, ** and * denote the significance at the 1%, 5% and 10% levels, respectively. LL is the log-likelihood value. AIC means the Akaike Information Criterion, while SC means the Schwarz Criterion. k is a constant which controls how many cumulative Fourier frequencies are included in the approximation of FFF-GARCH-type models. $Q(10)$, $Q(20)$ and $Q^2(10)$, $Q^2(20)$ denote the Ljung and Box statistics of the standardized residuals and squared residuals for up to 10th and 20th order serial autocorrelation, respectively.

Table A4: The results of the in-sample estimation of EGARCH model and FFF-EGARCH model for WTI crude oil market

	EGARCH	FFF-EGARCH				
		$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
δ	0.0779***(2.8299)	0.0776***(2.6191)	0.0770***(2.6048)	0.0768***(2.5997)	0.0747***(2.5280)	0.0766***(2.5914)
α_0	-0.0638***(-6.7558)	-0.0604***(-6.3098)	-0.0522***(-5.1122)	-0.0471***(-4.3642)	-0.0419***(-3.7250)	-0.0392***(-3.3860)
α_1	0.1100***(8.3352)	0.1092***(8.1508)	0.1081***(7.7567)	0.1085***(7.2361)	0.1070***(7.2601)	0.1073***(7.1560)
β_1	0.9882***(299.3286)	0.9864***(272.4337)	0.9818***(212.9183)	0.9785***(183.9605)	0.9759***(166.4239)	0.9741***(151.9646)
ξ	-0.0272***(-3.1757)	-0.0291***(-3.3345)	-0.0316***(-3.5026)	-0.0332***(-3.6073)	-0.0368***(-3.8378)	-0.0358***(-3.6589)
v	6.1286***(12.3557)	6.1495***(12.2135)	6.1679***(12.2633)	6.2193***(12.2385)	6.2428***(12.1760)	6.2774***(12.0684)
ϕ_1		0.0013(0.6687)	0.0017(0.7944)	0.0020(0.8692)	0.0022(0.9008)	0.0023(0.9233)
ψ_1		-1.6820*(0.0925)	-0.0048**(-2.0154)	-0.0057**(-2.2159)	-0.0063***(-2.3726)	-0.0067(-2.4171)***
ϕ_2			-0.0014(-0.6504)	-0.0017(-0.7108)	-0.0019(-0.7742)	-0.0022(-0.8814)
ψ_2			-0.0069**(-2.5917)	-0.0080***(-2.7613)	-0.0088***(-2.8562)	-0.0095***(-2.9185)
ϕ_3				0.0034*(1.4503)	0.0039*(1.6049)	0.0040*(1.5976)
ψ_3				-0.0042*(-1.6631)	-0.0045*(-1.6928)	-0.0049*(-1.7648)
ϕ_4					0.0042*(1.6336)	0.0045*(1.6776)
ψ_4					0.0028(1.1391)	0.0029(1.1424)
ϕ_5						-0.0040*(-1.4948)
ψ_5						-0.0004(-0.1840)

	EGARCH	FFF-EGARCH				
		$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
LL	-10016.8968	-10015.1031	-10010.5226	-10007.8943	-10005.7398	**-10004.3999**
AIC	4.3998	4.3999	4.3988	4.3985	**4.3984**	4.3987
SC	**4.4068**	4.4092	4.4104	4.4125	4.4148	4.4174
$Q(10)$	12.463[0.2742]	12.042[0.2822]	12.345[0.2626]	11.965[0.2873]	12.024[0.2834]	11.803[0.2984]
$Q(20)$	17.190[0.7047]	15.675[0.7365]	16.587[0.6795]	16.690[0.6730]	16.988[0.6537]	14.816[0.4647]
$Q^2(10)$	49.715[0.0000]	48.058[0.0000]	47.070[0.0000]	44.811[0.0000]	45.583[0.0000]	45.534[0.0000]
$Q^2(20)$	63.467[0.0000]	63.203[0.0000]	61.005[0.0000]	57.669[0.0000]	59.995[0.0000]	60.777[0.0000]

Note: The *t*-statistics values and *p*-values are reported in the parentheses and square brackets, respectively. ***, ** and * denote the significance at the 1%, 5% and 10% levels, respectively. LL is the log-likelihood value. AIC means the Akaike Information Criterion, while SC means the Schwarz Criterion. k is a constant which controls how many cumulative Fourier frequencies are included in the approximation of FFF-GARCH-type models. $Q(10)$, $Q(20)$ and $Q^2(10)$, $Q^2(20)$ denote the Ljung and Box statistics of the standardized residuals and squared residuals for up to 10th and 20th order serial autocorrelation, respectively.

Table A5: The results of the in-sample estimation of GJR-GARCH model and FFF-GJR-GARCH model for WTI crude oil market

	GJR-GARCH	FFF-GJR-GARCH				
		k=1	k=2	k=3	k=4	k=5
δ	0.0827***(2.7650)	0.0809***(2.7162)	0.0813***(2.7382)	0.0815***(2.7419)	0.0797***(2.6796)	0.0798***(2.6898)
α_0	0.0851***(3.9941)	0.1179***(4.2929)	0.1515***(4.4012)	0.1610***(4.4253)	0.1672***(4.4394)	0.1810***(4.4797)
α_1	0.0222***(1.9368)	0.0267***(2.2582)	0.0283***(2.3102)	0.0305***(2.4696)	0.0320***(2.5688)	0.0325***(2.5942)
β_1	0.9368***(111.6741)	0.9302***(95.3023)	0.9243***(83.2312)	0.9226***(79.9612)	0.9218***(78.4102)	0.9195***(75.2067)
ξ	0.0369***(4.2078)	0.0351***(3.9816)	0.0342(3.***7921)	0.0329***(3.6289)	0.0320***(3.5368)	0.0316***(3.4939)
v	6.1837***(12.1003)	6.2316***(11.9966)	6.2717***(11.9917)	6.2956***(11.9456)	6.3017***(11.9046)	6.3345***(11.8466)
ϕ_1		0.0102(0.8242)	0.0222*(1.4032)	0.0189(1.1538)	0.0160(0.9575)	0.0175***(0.9919)
ψ_1		-0.0405***(-2.9912)	-0.0405***(-2.8706)	-0.0389***(-2.6541)	-0.0401***(-2.7256)	-0.0459***(-2.9274)
ϕ_2			-0.0134(-1.0296)	-0.0127(-0.9063)	-0.0165(-1.1133)	-0.0182(-1.2038)
ψ_2			-0.0394***(-2.6172)	-0.0366***(2.3727)	-0.0426***(-2.5174)	-0.0458***(-2.5418)
ϕ_3				0.0079(0.5964)	0.0039(0.2855)	0.0084(0.5694)
ψ_3				-0.0196*(-1.4561)	-0.0206*(-1.4281)	-0.0153*(-0.9991)
ϕ_4					0.0177*(1.2997)	0.0291**(1.8946)
ψ_4					0.0096(0.7046)	0.0081(0.5657)
ϕ_5						-0.0247**(-1.1869)
ψ_5						-0.0099(-0.7069)

	GJR-GARCH	FFF-GJR-GARCH				
		$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
LL	-10018.9231	-10012.4717	-10007.4605	-10006.1121	-10004.9219	**-10002.9003**
AIC	4.4007	4.3988	**4.3974**	4.3977	4.3981	4.3981
SC	**4.4077**	4.4081	4.4091	4.4117	4.4144	4.4167
$Q(10)$	10.533[0.3950]	10.360[0.4095]	10.693[0.3819]	10.586[0.3906]	32.698[0.0003]	10.359[0.4095]
$Q(20)$	14.369[0.8113]	13.716[0.8446]	14.558[0.8011]	14.637[0.7968]	48.729[0.0003]	14.479[0.8054]
$Q^2(10)$	34.872[0.0004]	33.991[0.0001]	32.359[0.0003]	32.309[0.0003]	10.359[0.4095]	33.751[0.0002]
$Q^2(20)$	48.558[0.0004]	50.062[0.0002]	47.428[0.0005]	47.178[0.0005]	14.479[0.8054]	50.090[0.0002]

Note: The *t*-statistics values and *p*-values are reported in the parentheses and square brackets, respectively. ***, ** and * denote the significance at the 1%, 5% and 10% levels, respectively. LL is the log-likelihood value. AIC means the Akaike Information Criterion, while SC means the Schwarz Criterion. k is a constant which controls how many cumulative Fourier frequencies are included in the approximation of FFF-GARCH-type models. $Q(10)$, $Q(20)$ and $Q^2(10)$, $Q^2(20)$ denote the Ljung and Box statistics of the standardized residuals and squared residuals for up to 10th and 20th order serial autocorrelation, respectively.

Table A6: The results of likelihood ratio test

	FFF-GARCH model					FFF-EGARCH model					FFF-GJR-GARCH model				
	s=*k*=1	s=*k*=2	s=*k*=3	s=*k*=4	s=*k*=5	s=*k*=1	s=*k*=2	s=*k*=3	s=*k*=4	s=*k*=5	s=*k*=1	s=*k*=2	s=*k*=3	s=*k*=4	s=*k*=5
WTI	0.0209	0.0436	0.7345	0.7705	0.4157	0.4647	0.0472	0.2619	0.3657	0.6127	0.0117	0.0400	0.6097	0.6661	0.4002
Brent	0.0174	0.0189	0.5486	0.7811	0.4515	0.3164	0.0055	0.1100	0.2324	0.3442	0.0060	0.0045	0.3134	0.6125	0.5284

Note: The numbers in the table are the *p*-values of likelihood ratio test. The likelihood ratio test is that we construct Chi-square statistics $\chi^2 = 2*(LL_A - LL_B)$ to test the null hypothesis $H_0: LL_A = LL_B$, and it indicates the fitting performance of model A is superior to model B when the null hypothesis is rejected, where LL_A is the log-likelihood value of model A, and LL_B is the log-likelihood value of model B; *k* is a constant which controls how many cumulative Fourier frequencies are included in the approximation of FFF-GARCH-type models, and *s* can identify two compared models. Specifically, *s*=*k* indicates that model A is the FFF-GARCH-type model with the Fourier frequency *k*, and model B is the FFF-GARCH-type model with *k*-1. It should be noted that *k*=0 indicates the GARCH-type models without trigonometric terms.

Table A7: The 10-day ahead forecasting results for crude oil volatility

		*MSE*1	*MSE*2	*MAE*1	*MAE*2	*QLIKE*	R^2LOG	*MAPE*
Panel A*: *The 10-day ahead forecasting results for Brent crude oil volatility								
GARCH		4.4447	1196.6440	1.6196	22.0728	4.4343	0.5245	39.0403
FFF-GARCH	$k=1$	**4.3117**	1182.3797	**1.5789**	**21.4091**	4.4355	**0.5041**	**37.6511**
	$k=2$	4.4312	1194.8164	1.5983	21.6455	4.4402	0.5211	37.8451
	$k=3$	4.4879	1195.4062	1.6035	21.8105	4.4383	0.5244	38.3989
	$k=4$	4.7000	1228.8515	1.6410	22.3930	4.4439	0.5405	38.9440
	$k=5$	4.4735	**1179.5935**	1.5882	21.7892	**4.4310**	0.5128	39.1692
EGARCH		4.0159	1210.6342	1.4346	19.9006	4.4237	0.4112	35.0008
FFF-EGARCH	$k=1$	4.0530	1204.0494	**1.4325**	19.7330	4.4327	**0.4107**	33.8403
	$k=2$	4.1004	1206.0814	1.4371	19.7478	4.4333	0.4143	**33.5080**
	$k=3$	**4.0037**	1156.2027	1.4352	**19.6853**	**4.4259**	0.4158	34.4130
	$k=4$	4.1006	1165.2528	1.4646	19.9743	4.4340	0.4327	34.5592
	$k=5$	4.0710	**1127.9266**	1.4573	19.8865	4.4386	0.4323	35.1086
GJR-GARCH		3.7367	1023.3091	1.4782	19.8785	4.4103	0.4592	37.4532
FFF-GJR-GARCH	$k=1$	**3.6837**	**1012.6075**	1.4704	**19.6949**	**4.4095**	0.4585	37.1223
	$k=2$	3.9277	1137.9342	**1.4548**	19.7470	4.4180	**0.4443**	**35.3045**
	$k=3$	3.8910	1114.3185	1.4513	19.7306	4.4137	0.4436	35.9050
	$k=4$	4.0810	1142.1510	1.4912	20.2536	4.4213	0.4613	36.2187
	$k=5$	4.0519	1053.4228	1.5349	20.7810	4.4154	0.4915	38.7808
MRS-GARCH		4.5067	1275.2684	1.6033	22.1079	4.4250	0.5094	39.0366
Panel B*:*The 10-day ahead forecasting results for WTI crude oil volatility								
GARCH		6.2148	2394.8202	1.8518	27.8987	4.6251	0.5894	47.1294
FFF-GARCH	$k=1$	**5.8769**	2339.2357	**1.7516**	**26.3996**	4.6195	0.5330	45.0979
	$k=2$	6.0193	2351.6060	1.7783	26.7232	4.6233	0.5408	44.9774
	$k=3$	6.0216	2334.4218	1.7813	26.8041	4.6199	0.5433	45.4167

		$MSE1$	$MSE2$	$MAE1$	$MAE2$	$QLIKE$	R^2LOG	$MAPE$
	$k=4$	6.2130	2365.6751	1.8175	27.2873	4.6292	0.5639	45.6662
	$k=5$	5.9039	2287.6678	1.7551	26.4836	4.6135	0.5326	45.4246
EGARCH		5.51891	2942.8234	1.5496	24.4908	4.5970	0.4085	42.6713
	$k=1$	5.2787	2547.1496	1.5291	23.8286	4.6019	0.3990	41.2600
	$k=2$	5.1847	2274.1580	1.5204	23.3223	4.6052	0.3980	39.9729
FFF-EGARCH	$k=3$	5.0585	2214.6751	1.5101	23.1369	4.5968	0.3962	40.6948
	$k=4$	5.3538	2321.6673	1.5548	23.7712	4.6109	0.4188	40.1929
	$k=5$	5.2963	2502.5617	1.5392	23.8051	4.6046	0.4086	40.6855
GJR-GARCH		5.5383	2090.5029	1.7776	26.3118	4.6060	0.5631	47.1825
	$k=1$	5.1311	2070.0496	1.6373	24.3855	4.5925	0.4884	44.6605
	$k=2$	7.2683	2922.5682	1.8343	27.6558	4.6914	0.5794	40.8217
FFF-GJR-GARCH	$k=3$	5.2971	2097.7633	1.6650	24.8034	4.5925	0.4995	44.9905
	$k=4$	5.5360	2288.9781	1.6290	24.4186	4.6099	0.4815	42.4102
	$k=5$	5.3497	2240.3782	1.5917	23.9471	4.5995	0.4626	42.4671
MRS-GARCH		6.8824	2769.7299	1.8766	29.5505	4.6083	0.5491	47.3099

Note: The numbers in the table refer to the values of loss functions. Underlined numbers indicate that the corresponding models have the lowest forecasting losses under a prespecified criterion among six models (i.e., the model without trigonometric terms and those with different trigonometric frequencies).

Table A8: The 22-day ahead forecasting results for crude oil volatility

		*MSE*1	*MSE*2	*MAE*1	*MAE*2	*QLIKE*	R^2LOG	*MAPE*
***Panel A*: The 22-day ahead forecasting results for Brent crude oil volatility**								
GARCH		7.0428	3985.5130	2.06239	41.6684	5.2515	0.3631	89.5717
FFF-GARCH	k =1	6.8137	3867.4648	1.9954	39.6311	5.2541	0.3482	85.8394
	k =2	7.1318	3941.8446	2.0584	40.5535	5.2597	0.3654	86.3602
	k =3	7.2331	3952.4097	2.0849	41.3655	5.2571	0.3669	87.8764
	k =4	7.8025	4184.6482	2.1800	43.6711	5.2643	0.3851	89.3340
	k =5	7.06331	3849.0846	2.0547	41.4508	5.2466	0.3480	89.9485
EGARCH		8.8225	5001.5198	2.1205	42.2210	5.3546	0.3865	68.5377
FFF-EGARCH	k =1	9.4896	5283.7363	2.2026	43.6406	5.3762	0.4111	66.4846
	k =2	9.3867	5346.5045	2.1603	43.4379	5.3550	0.3837	66.8034
	k =3	8.2081	4790.8480	2.0141	40.8484	5.3139	0.3376	70.0899
	k =4	8.2281	4755.8592	2.0139	40.8105	5.3155	0.3406	70.6450
	k =5	7.6022	4307.9501	1.9238	38.8610	5.3082	0.3264	72.7387
GJR-GARCH		5.3678	3193.5699	1.7330	34.3667	5.2238	0.2732	81.8912
FFF-GJR-GARCH	k =1	4.9403	2814.8624	1.7224	33.5591	5.2162	0.2818	84.9127
	k =2	6.6201	4113.3700	1.8644	37.8046	5.2464	0.2838	75.8686
	k =3	6.2809	3910.9108	1.8290	37.1338	5.2375	0.2731	77.3387
	k =4	6.6496	4009.0066	1.8934	38.3910	5.2477	0.2920	78.1035
	k =5	5.7287	3041.1152	1.9090	37.7574	5.2237	0.3104	89.3709
MRS-GARCH		7.5723	4548.0996	2.0742	42.0854	5.2481	0.3725	88.7815
***Panel B*: The 22-day ahead forecasting results for WTI crude oil volatility**								
GARCH		12.2163	9591.0052	2.6851	60.2511	5.4616	0.4785	107.2698
FFF-GARCH	k =1	11.4592	9170.2285	2.4927	55.5816	5.4586	0.4244	101.6864
	k =2	11.9140	9256.0552	2.5510	56.7897	5.4647	0.4344	101.3982
	k =3	11.8584	9149.0162	2.5659	57.2340	5.4598	0.4351	102.6518

		$MSE1$	$MSE2$	$MAE1$	$MAE2$	$QLIKE$	R^2LOG	$MAPE$
	k =4	12.4480	9383.8240	2.6601	59.1160	5.4732	0.4624	103.3247
	k =5	11.3971	8816.6048	2.5078	56.1895	5.4505	0.4190	103.2122
EGARCH		13.8036	11646.9355	2.4418	54.7278	5.6072	0.4735	80.2153
	k =1	14.1396	11136.2114	2.4985	55.4730	5.6261	0.4900	77.8659
	k =2	14.2669	11154.3271	2.4880	55.5596	5.6134	0.4759	76.6397
FFF-EGARCH	k =3	13.2422	10596.2182	2.3806	53.5068	5.5755	0.4363	78.8007
	k =4	13.8567	11057.0796	2.4382	55.0493	5.5857	0.4466	78.2820
	k =5	12.9220	10432.0837	2.3496	53.1162	5.5631	0.4236	80.1524
GJR-GARCH		10.1107	7663.42217	2.5050	54.9172	5.4342	0.4358	107.6330
	k =1	9.1414	7449.7647	2.2545	49.7198	5.4196	0.3598	100.8325
	k =2	15.1772	12059.4235	2.5746	57.7435	5.5831	0.4812	86.1915
FFF-GJR-GARCH	k =3	9.4975	7568.1842	2.3125	51.1706	5.4192	0.3677	101.8710
	k =4	11.4496	9582.4754	2.2926	51.768	5.4731	0.3735	89.7561
	k =5	10.8358	9242.7350	2.2020	50.1612	5.4558	0.3494	90.1591
MRS-GARCH		12.3370	10186.0138	2.5801	60.4306	5.4681	0.4075	106.2032

Note: The numbers in the table refer to the values of loss functions. Underlined numbers indicate that the corresponding models have the lowest forecasting losses under a prespecified criterion among six models (i.e., the model without trigonometric terms and those with different trigonometric frequencies).

Table A9: The 1-day ahead volatility forecasting results for daily data under new sample

		*MSE*1	*MSE*2	*MAE*1	*MAE*2	*QLIKE*	R^2LOG	*MAPE*
***Panel A*: The 1-day ahead forecasting results for Brent crude oil volatility**								
GARCH		2.8783	118.4281	1.3807	6.3469	2.7419	<u>7.3158</u>	3.8062
FFF-GARCH	*k* =1	<u>2.8212</u>	<u>118.1469</u>	<u>1.3669</u>	<u>6.2103</u>	2.7317	7.3759	<u>3.6747</u>
	k =2	2.9552	118.9733	1.4097	6.4463	<u>2.7240</u>	7.5689	3.8473
	k =3	2.8815	118.5742	1.3849	6.3268	2.7335	7.4171	3.7601
	k =4	2.8680	118.4004	1.3799	6.3062	2.7299	7.3881	3.7589
	k =5	2.8267	118.2843	1.3682	6.2128	2.7331	7.4017	3.6657
EGARCH		2.9062	119.6269	1.3880	6.4581	2.7910	7.3032	3.9080
FFF-EGARCH	*k* =1	2.7176	117.2916	1.3422	6.1073	2.7153	7.1611	3.6005
	k =2	2.7438	<u>116.1038</u>	<u>1.3123</u>	<u>5.9035</u>	2.7110	7.1899	3.6524
	k =3	2.7191	116.5033	1.3388	6.0763	<u>2.7043</u>	<u>7.1659</u>	3.5359
	k =4	<u>2.7102</u>	119.3805	1.3149	5.9471	2.7740	7.0690	3.3598
	k =5	2.7130	121.0104	1.3519	6.1548	2.7990	7.3742	<u>3.2693</u>
GJR-GARCH		2.7648	116.1590	1.3599	6.2083	2.7459	7.2268	3.9453
FFF-GJR-GARCH	*k* =1	<u>2.7107</u>	115.5351	<u>1.3026</u>	<u>5.9902</u>	<u>2.6953</u>	<u>7.1165</u>	3.5152
	k =2	2.7479	<u>115.5217</u>	1.3521	6.1326	2.7098	7.2603	3.0955
	k =3	2.7501	115.7353	1.3517	6.1396	2.7125	7.2248	3.1895
	k =4	2.7429	116.4693	1.3303	6.0259	2.7670	7.1410	5.1856
	k =5	2.7258	116.3279	1.3447	6.1041	2.7032	7.2012	<u>3.0732</u>
MRS-GARCH		2.9022	117.2463	1.3945	6.4290	2.7073	7.3622	3.9204
***Panel B*:The 1-day ahead forecasting results for WTI crude oil volatility**								
GARCH		2.9422	151.0018	<u>1.3547</u>	6.8488	2.7395	6.1535	4.6550
FFF-GARCH	*k* =1	<u>2.8681</u>	<u>149.6738</u>	1.3585	<u>6.7390</u>	<u>2.7453</u>	6.1670	<u>4.4633</u>
	k =2	2.9770	150.7418	1.3842	6.9132	2.7485	6.2553	4.6520
	k =3	2.9184	150.3822	1.3623	6.8120	2.7447	<u>6.1308</u>	4.5716

		$MSE1$	$MSE2$	$MAE1$	$MAE2$	$QLIKE$	R^2LOG	$MAPE$
	$k=4$	2.9396	150.4378	1.3720	6.8490	2.7462	6.2072	4.6032
	$k=5$	2.8776	149.7712	1.3639	6.7622	2.7451	6.1861	4.4663
EGARCH		2.9908	151.5217	1.3552	6.9387	2.7402	6.0253	4.9754
FFF-EGARCH	$k=1$	<u>2.8890</u>	<u>150.1019</u>	1.3419	6.6812	2.7731	<u>5.9493</u>	4.4777
	$k=2$	2.8306	149.2797	<u>1.3192</u>	<u>6.6535</u>	2.7351	6.0658	4.4973
	$k=3$	2.9233	174.0723	1.3345	6.7342	2.7411	5.9963	<u>4.3814</u>
	$k=4$	2.9080	151.3256	1.3543	6.8060	<u>2.7283</u>	6.1351	4.5523
	$k=5$	2.8947	160.4923	1.3242	6.4660	3.2671	6.2085	4.7933
GJR-GARCH		2.8479	148.5359	1.3511	6.7363	2.7391	6.0570	4.5940
FFF-GJR-GARCH	$k=1$	<u>2.8079</u>	<u>148.4172</u>	<u>1.3433</u>	<u>6.6341</u>	2.7351	<u>5.9855</u>	4.3651
	$k=2$	2.8594	148.5269	1.3599	6.7553	2.7348	6.1203	4.5072
	$k=3$	2.8352	148.9897	1.3468	6.7030	2.7404	6.0258	4.4519
	$k=4$	2.8431	148.6132	1.3553	6.7308	<u>2.7321</u>	6.1106	4.4935
	$k=5$	2.8161	149.6196	1.3470	6.6717	2.7349	6.0726	<u>4.3644</u>
MRS-GARCH		2.9717	152.7225	1.3894	6.8390	2.7522	6.7771	4.6641

Note: The numbers in the table refer to the values of loss functions. Underlined numbers indicate that the corresponding models have the lowest forecasting losses under a prespecified criterion among six models (i.e., the model without trigonometric terms and those with different trigonometric frequencies).

Table A10: The 1-week ahead volatility forecasting results for weekly data

		*MSE*1	*MSE*2	*MAE*1	*MAE*2	*QLIKE*	R^2LOG	*MAPE*
***Panel A*: The 1-week ahead forecasting results for Brent crude oil volatility**								
GARCH		7.9318	1014.1275	2.1564	16.7566	3.4660	8.9988	244.5733
	k =1	7.5650	1005.3861	2.0946	16.1687	3.4590	8.8120	229.0731
	k =2	7.6311	1013.1246	2.1168	16.3298	3.4646	8.8428	212.0364
FFF-GARCH	k =3	7.7561	1012.5583	2.1378	16.5687	3.4588	8.8748	213.8519
	k =4	8.1195	1028.5880	2.1817	17.0909	3.4729	8.9426	289.7845
	k =5	7.7700	1007.6381	2.1207	16.5404	3.4490	8.7915	277.4971
EGARCH		8.0069	998.0400	2.1687	17.3323	3.4080	8.9088	263.0069
	k =1	7.9318	1004.1275	2.1564	16.9983	3.2435	8.9988	271.9318
	k =2	7.3197	995.6378	2.1486	16.7566	3.3660	8.8954	242.2735
FFF-EGARCH	k =3	7.5861	1014.9452	2.1601	17.2361	3.4136	8.8144	260.3335
	k =4	7.3781	1009.2635	2.1696	17.5742	3.4560	8.8632	259.3964
	k =5	7.5138	1026.8532	2.1726	17.9764	3.2695	8.8387	263.2954
GJR-GARCH		7.6906	964.1318	2.1611	16.6938	3.3474	9.0746	275.7120
	k =1	6.9161	933.3836	2.0401	15.6027	3.7979	8.6774	217.8359
	k =2	6.8331	929.6210	2.0548	15.6447	3.7324	8.6696	204.9714
FFF-GJR-GARCH	k =3	6.9291	923.7754	2.0743	15.8237	3.7868	8.7280	214.0948
	k =4	7.3145	937.9689	2.1275	16.3624	3.4216	8.8447	217.0952
	k =5	7.2067	954.6082	2.0596	15.8927	3.9279	8.6809	238.4732
MRS-GARCH		7.5811	1010.9231	2.1123	15.9111	3.4979	8.9126	215.2483
***Panel B*: The 1-week ahead forecasting results for WTI crude oil volatility**								
GARCH		6.6818	613.7182	2.0820	14.9263	5.0181	6.4767	142.6115
	k =1	6.4081	612.3462	2.0166	14.4421	4.9104	6.2403	126.9099
FFF-GARCH	k =2	6.5587	624.5300	2.0410	14.7062	5.2020	6.2018	113.9125
H	k =3	6.6091	620.1714	2.0527	14.8260	5.1392	6.2567	118.1202

		$MSE1$	$MSE2$	$MAE1$	$MAE2$	$QLIKE$	R^2LOG	$MAPE$
	k =4	6.7683	626.1900	2.0750	15.0730	5.3010	6.3113	123.3149
	k =5	6.5353	610.1143	2.0246	14.7040	4.9714	6.2836	116.7165
EGARCH		7.6269	685.8374	2.0817	15.9947	4.9249	8.0570	139.6089
FFF-EGARCH	k =1	7.8723	694.8733	1.9978	<u>14.7135</u>	4.5435	8.0478	120.3980
	k =2	7.2257	<u>673.9493</u>	<u>1.9791</u>	14.8451	4.3160	7.5234	<u>117.1735</u>
	k =3	<u>6.9728</u>	683.2849	1.9471	15.6514	<u>4.9848</u>	<u>7.4849</u>	126.8325
	k =4	7.9178	696.0594	2.1974	15.7278	4.4551	7.8054	123.9169
	k =5	7.3863	690.8532	2.1613	14.9764	4.5935	7.7957	119.1352
GJR-GARCH		6.4766	584.4140	2.0704	14.7477	4.9151	6.4841	150.9886
FFF-GJR-GARCH	k =1	<u>5.9723</u>	578.5564	<u>1.8743</u>	<u>14.0458</u>	<u>4.5642</u>	<u>6.0310</u>	131.6879
	k =2	6.0081	580.6211	1.9809	14.1839	4.6585	6.1187	<u>117.0604</u>
	k =3	6.0771	<u>573.4248</u>	1.9988	14.3148	4.6768	6.0965	123.7845
	k =4	6.2781	588.1221	2.0359	14.6609	4.8835	6.1755	131.9520
	k =5	6.1003	583.9793	1.9893	14.3044	4.6639	6.0881	128.2972
MRS-GARCH		6.5752	635.7162	2.1038	14.9727	4.9581	6.3772	138.9433

Note: The numbers in the table refer to the values of loss functions. Underlined numbers indicate that the corresponding models have the lowest forecasting losses under a prespecified criterion among six models (i.e., the model without trigonometric terms and those with different trigonometric frequencies).

Appendix B

The evaluation criteria for forecasting performance

According to Hansen and Lunde (2005), we apply six widely used statistical loss functions to evaluate the out-of-sample forecasting performance for crude oil price volatility, which are defined as Eqs. (B1)-(B7).

$$MAE1 = \frac{1}{T-N}\sum_{t=N+1}^{T} | \sqrt{\hat{h}_t} + \sqrt{h_t} | \quad \text{(B1)}$$

$$MAE2 = \frac{1}{T-N}\sum_{t=N+1}^{T} | \hat{h}_t - h_t | \quad \text{(B2)}$$

$$MSE1 = \frac{1}{T-N}\sum_{t=N+1}^{T} (\sqrt{\hat{h}_t} - \sqrt{h_t})^2 \quad \text{(B3)}$$

$$MSE2 = \frac{1}{T-N}\sum_{t=N+1}^{T} (\hat{h}_t - h_t)^2 \quad \text{(B4)}$$

$$QLIKE = \frac{1}{T-N}\sum_{t=N+1}^{T} (\ln(\hat{h}_t) - h_t / \hat{h}_t)^2 \quad \text{(B5)}$$

$$R^2LOG = \frac{1}{T-N}\sum_{t=N+1}^{T} (\ln(h_t / \hat{h}_t))^2 \quad \text{(B6)}$$

$$MAPE = \frac{1}{T-N}\sum_{t=N+1}^{T} | 1 - \hat{h}_t / h_t | \quad \text{(B7)}$$

where h_t represents the actual volatility whereas $\hat{h}_t$ represents the forecasted volatility; T and N respectively stand for the number of full-sample and in-sample observations, while $T-N$ is the out-of-sample observations. Following Kang et al. (2009) and Wei et al. (2010), daily actual volatility is assessed by daily squared returns (r_t^2).

Economic significance

Assuming that a mean-variance investor who optimally allocates between crude oil and risk-free bills based on the various volatility forecasts, and the utility U_t of portfolio strategy P is defined as follows:

$$\begin{aligned} U_t(R_t^P) &= E_t(R_t^P) - 0.5\gamma Var_t(R_t^P) \\ &= \omega_t(r_t^e + r_t^f) + (1-\omega_t)r_t^f - 0.5\gamma\omega_t^2\sigma_t^2 \end{aligned} \quad \text{(B8)}$$

where $E_t(.)$ and $Var_t(.)$ represent the conditional mean and conditional variance of the portfolio return R_t^P at time *t*, respectively. r_t^e and σ_t^2 denote crude oil excess return and volatility on day *t*, respectively; r_t^f is risk-free rate; γ is the investor's coefficient of relative risk aversion[1]; and ω_t means the portfolio weight. Maximizing the objective function, we can obtain the optimal portfolio weight ω_t^* given by:

$$\omega_t^* = \frac{1}{\gamma}\left(\frac{\hat{r}_{t+1}^e}{\hat{\sigma}_{t+1}^2}\right) \tag{B9}$$

where $\hat{r}_{t+1}^e$ and $\hat{\sigma}_{t+1}^2$ are the out-of-sample forecasting value of excess return and volatility of crude oil price, respectively. Specifically, we use the prevailing historical average to forecast $\hat{r}_{t+1}^e$ (Campbell and Thompson, 2008), and we use the forecasting models above to ensure the value of $\hat{\sigma}_{t+1}^2$. In addition, we restrict the value of ω_t^* to lie between 0 and 1.5 because of short-sale constraint. Therefore, we compute the portfolio return R_{t+1}^P as:

$$R_{t+1}^P = \omega_t^*(r_{t+1}^e + r_{t+1}^f) + (1-\omega_t^*)r_{t+1}^f \tag{B10}$$

This paper also employs two indicators to measure the performance of the portfolio, namely certainty equivalent return (CER) and Sharpe ratio (SR), which is given by:

$$CER = \hat{\mu}_p - \frac{\gamma}{2}\hat{\sigma}_p^2 \tag{B11}$$

$$SR = \frac{\hat{\mu}_p}{\hat{\sigma}_p^2} \tag{B12}$$

where $\hat{\mu}_p$ and $\hat{\sigma}_p^2$ are the sample mean and variance, respectively, of the portfolio returns over the out-of-sample evaluation period.

[1] To make the results more robust, we set the value of γ as 3, 6 and 9 (Zhang et al., 2019b).

Appendix C

Robustness checks

We use the DM statistics to test the forecasting ability of different models under different forecasting horizon, which is mainly based on two loss functions, i.e., MSE2 and MAE2. The test results are shown in Table C1. Specifically, the null hypothesis of DM test is that the predictive abilities of two models are equal (Diebold and Mariano, 1995). Therefore, if the null hypothesis is rejected, it means that there is significant difference between their forecasting errors. The results are as follows:

From the values of DM test, we can find that the model considering smooth shifts exhibits better out-of-sample forecasting performance than the benchmark model. Specifically, according to the results of panel A of Table C1, compared with the benchmark model (i.e., GARCH model), most of the DM test results of FFF-GARCH-type models significantly reject the null hypothesis. It indicates that the forecasting performance of the model considering smooth shifts is obviously better than GARCH model. Meanwhile, compared with the benchmark model (MRS-GARCH model) in panel B, the DM test results of the FFF-GARCH-type models significantly reject the null hypothesis, which indicates that the model considering the smooth shifts exhibit better forecasting performance than the that considering the regime switching in term of crude oil price volatility forecasting.

Table C1: The DM test results of different volatility forecasting models

		h=1		*h*=5		*h*=10		*h*=22	
		MSE2	*MAE2*	*MSE2*	*MAE2*	*MSE2*	*MAE2*	*MSE2*	*MAE2*
***Panel A*: GARCH model as a benchmark model (Brent)**									
EGARCH		0.8125**	0.1108*	10.4460**	-0.6234*	13.9902**	-2.1722***	1016.0068***	0.5526**
GJR-GARCH		-1.6898**	-0.0523***	-42.7080***	-0.7465**	-173.3349**	-2.1943***	-791.9431***	-7.3017***
MRS-GARCH		1.5795	0.0611	26.1072***	0.0791	78.6244	0.0351	562.5866***	0.4170***
FFF-GARCH	*k* =1	-0.1853***	-0.1145***	-2.6597*	-0.3877*	-14.2643***	-0.6637***	-118.0482**	-2.0373***
FFF-EGARCH	*k* =1	-0.5983	-0.0240**	-7.8100*	-1.0024***	7.4054***	-2.3398***	1298.2233***	1.9722***
FFF-GJR-GARCH	*k* =1	-1.4609***	-0.1870***	-29.9641***	-1.0498***	-184.0365***	-2.3779***	-1170.6506***	-8.1093***
***Panel B*: MRS-GARCH model as a benchmark model (Brent)**									
EGARCH		-0.7670	0.0497*	-15.6612**	-0.7025**	-64.6342**	-2.2073***	453.4202***	0.1356

		h=1		*h*=5		*h*=10		*h*=22	
		MSE2	*MAE2*	*MSE2*	*MAE2*	*MSE2*	*MAE2*	*MSE2*	*MAE2*
GJR-GARCH		-3.2693***	-0.1134***	-68.8152**	-0.8256	-251.9593**	-2.2294***	-1354.5297***	-7.7187***
FFF-GARCH	*k* =1	-1.7648	-0.1756***	-28.7669*	-0.4668**	-92.8887***	-0.6988**	-680.6348***	-2.4543***
FFF-EGARCH	*k* =1	-2.1778***	-0.0851	-33.9172***	-1.0815***	-71.219**	-2.3749***	735.6367***	1.5552***
FFF-GJR-GARCH	*k* =1	-3.0404***	-0.1134**	-66.0713***	-1.1289***	-262.6609***	-2.4130***	-1733.2372***	-8.5263***

Note: The numbers in the table refer to the difference of the loss functions of the benchmark model and comparison model, i.e., *MSE2(L1)-MSE2(L2)* or *MAE2(L1)-MAE2(L2)*, where *L1* and *L2* represent the comparison model and benchmark model, respectively. ***, ** and * denote the significance at the 1%, 5% and 10% levels, respectively.