## 7. APPENDIX

### 7.1 Proof of Lemmata, Propositions, and Corollary

*Proof.* We show that $g_{n,1} = g_{m,1}, \forall n, m = 1, \ldots, N$. The result that $g_{n,2} = g_{m,2}, \forall n, m = 1, \ldots, N$ can be shown analogously and is excluded for brevity. Because of the assumed properties of the inverse demand and cost functions, each generator's profit-maximisation problem is convex. Thus, the Karush-Kuhn-Tucker (KKT) conditions for $g_{n,1}$, which are:

$$-A_1 + Z_1 \cdot (g_1^G - Fd) + Z_1 g_{n,1} + c'(g_{n,1}) - \mu_{n,1} = 0 \tag{17}$$

$$0 \leq \mu_{n,1} \perp g_{n,1} \geq 0, \tag{18}$$

where $\mu_{n,1}$ is the Lagrange multiplier associated with generator $n$'s period-1 non-negativity constraint, are sufficient for a global optimum.

Subtracting condition (17) for generator $m$ from that for generator $n \neq m$ gives:

$$Z_1 \cdot (g_{n,1} - g_{m,1}) + c'(g_{n,1}) - c'(g_{m,1}) - \mu_{n,1} + \mu_{m,1} = 0. \tag{19}$$

Suppose for contradiction that the production levels are not symmetric and without loss of generality that the generators are labelled such that $g_{n,1} > g_{m,1}$. From condition (18) we must have $\mu_{n,1} = 0$. Thus, (19) becomes:

$$Z_1 \cdot (g_{n,1} - g_{m,1}) + c'(g_{n,1}) - c'(g_{m,1}) + \mu_{m,1} = 0,$$

which cannot hold because by assumption we have that $Z_1 \cdot (g_{n,1} - g_{m,1}) > 0$, by convexity of the cost function we have that $c'(g_{n,1}) - c'(g_{m,1}) \geq 0$, and the KKT conditions require that $\mu_{m,1} \geq 0$. This gives the desired contradiction, which proves the result. ■

*Proof.* Assume for contradiction that $(k^*, d^*)$ is an optimal solution in which $k^* \neq d^*$. By the inequality constraint in the problem, we must have $k^* > d^*$. Consider the alternate solution $(\tilde{k}, d^*)$, with:

$$\tilde{k} = \frac{k^* + d^*}{2}.$$

This solution is clearly feasible in the problem constraint. Moreover, we have that:

$$[\mathcal{P}(d^*) - (Ik^{*2})/2] - [\mathcal{P}(d^*) - (I\tilde{k}^2)/2] = I \cdot (\tilde{k}^2 - k^{*2})/2 < 0,$$

because by construction $\tilde{k} < k^*$, meaning that $(\tilde{k}, d^*)$ gives a smaller objective-function value than $(k^*, d^*)$, contradicting the optimality of $(k^*, d^*)$. ■

*Proof.* To show the first part of the proposition, we compare the expressions in (7) and (12), which gives that for $k_W^*$ to be greater than or equal to $k_\Pi^*$ we must have:

$$\frac{A_2 - FA_1 - BN \cdot (F-1)(N+2)}{I \cdot (N+1)^2 + F^2 Z_1 + Z_2} \geq \frac{A_2 - FA_1 - BN \cdot (F-1)}{I \cdot (N+1) + 2F^2 Z_1 + 2Z_2},$$

which simplifies to:

$$Q(N) = -N^2 \cdot \Big[ I \cdot (A_2 - FA_1) + IB \cdot (F-1) + 2(F^2 Z_1 + Z_2) B \cdot (F-1) \Big]$$
$$-N \cdot \Big[ I \cdot (A_2 - FA_1) + IB \cdot (F-1) + 3(F^2 Z_1 + Z_2) B \cdot (F-1) \Big] + (A_2 - FA_1)(F^2 Z_1 + Z_2) \geq 0.$$

Because the coefficients of $N$ and $N^2$ in $Q(N)$ are both negative, we have $Q''(N) < 0, \forall N$ and $Q'(0) < 0$. Thus, $Q(N)$ is a downward-facing parabola. Furthermore, because $Q(0) > 0$, $Q(N)$

has a unique positive root, $\bar{N}$. Hence, for $N < \bar{N}$, we have $k_W^* \geq k_\Pi^*$, otherwise, $k_W^* < k_\Pi^*$.

To show the second part of the proposition, we note that from (10) we have that:

$$W'(k) = F\frac{Z_1}{(N+1)^2}\left[N\frac{A_1 - B}{Z_1} + (2N+1)Fk\right] + \frac{Z_2}{(N+1)^2}\left[(2N+1)k - N\frac{A_2 - B}{Z_2}\right] + (k_\Pi^* - k)\frac{I \cdot (N+1) + 2F^2 Z_1 + 2Z_2}{N+1}. \tag{20}$$

Substituting $k_\Pi^*$ into (20) gives:

$$\begin{aligned} W'(k_\Pi^*) &= F\frac{Z_1}{(N+1)^2}\left[N\frac{A_1 - B}{Z_1} + (2N+1)Fk_\Pi^*\right] + \frac{Z_2}{(N+1)^2}\left[(2N+1)k_\Pi^* - N\frac{A_2 - B}{Z_2}\right] \\ &= \frac{Q(N)}{(N+1)^2[I \cdot (N+1) + 2(F^2 Z_1 + Z_2)]}. \end{aligned}$$

The denominator, $(N+1)^2[I \cdot (N+1) + 2(F^2 Z_1 + Z_2)]$, is strictly positive. Thus, the only way for $W'(k_\Pi^*)$ to be positive (negative) is if $N < \bar{N}$ $(N > \bar{N})$. ■

*Proof.* $\bar{N}$ is defined as the root of the characteristic polynomial (*cf.* Proposition 1):

$$Q(\bar{N}) = 0.$$

To show the first part of the proposition, we totally differentiate this defining equation with respect to $I$, which gives:

$$\frac{\partial}{\partial I}Q(\bar{N}) + \frac{\partial}{\partial N}Q(\bar{N})\frac{\partial \bar{N}}{\partial I} = 0.$$

This can be rewritten as:

$$\frac{\partial \bar{N}}{\partial I} = -\frac{\frac{\partial}{\partial I}Q(\bar{N})}{\frac{\partial}{\partial N}Q(\bar{N})}. \tag{21}$$

We have:

$$\frac{\partial}{\partial I}Q(\bar{N}) = -\left(\bar{N}^2 + \bar{N}\right)[A_2 - FA_1 + B \cdot (F-1)] < 0,$$

and we also know (*cf.* the proof of Proposition 1) that $Q''(N) < 0, \forall N$ and $Q'(0) < 0$, meaning that:

$$\frac{\partial}{\partial N}Q(\bar{N}) < 0.$$

Thus, from (21) we have that:

$$\frac{\partial \bar{N}}{\partial I} < 0,$$

which is the desired result.

To show the second part of the proposition, we totally differentiate $Q(\bar{N}) = 0$ with respect to $B$, which gives:

$$\frac{\partial}{\partial B}Q(\bar{N}) + \frac{\partial}{\partial N}Q(\bar{N})\frac{\partial \bar{N}}{\partial B} = 0,$$

and which can be rewritten as:

$$\frac{\partial \bar{N}}{\partial B} = -\frac{\frac{\partial}{\partial B}Q(\bar{N})}{\frac{\partial}{\partial N}Q(\bar{N})}.$$

We have that:

$$\frac{\partial}{\partial B}Q(\bar{N}) = -I \cdot (F-1)\bar{N} \cdot \left(\overline{N} + 1\right) - \left(F^2 Z_1 + Z_2\right)(F-1)N \cdot (2N+3) < 0,$$

and we know that:

$$\frac{\partial}{\partial N} Q(\bar{N}) < 0.$$

Thus, we have:

$$\frac{\partial \bar{N}}{\partial B} < 0,$$

which is the desired result. ■

*Proof.* To show the impact of storage use on the price differential, we note that from (3) and (4) we have:

$$p_2(d) - F p_1(d) = \frac{A_2 - F A_1 - BN \cdot (F-1) - d \cdot (Z_2 + F^2 Z_1)}{N+1}. \tag{22}$$

The coefficient on $d$:

$$-\frac{Z_2 + F^2 Z_1}{N+1},$$

is negative, meaning that the price differential decreases with storage use.

To show the impact of the number of generating firms on the price differential, we partially differentiate (22) with respect to $N$, which gives:

$$\frac{\partial}{\partial N}(p_2(d) - F p_1(d)) = -\frac{[p_2(d) - F p_1(d) + B \cdot (F-1)]}{N+1}.$$

This partial derivative is negative. Thus, it follows that the price differential decreases with $N$. ■

*Proof.* To, first, show the effect of the number of generating firms on the profit maximiser's storage-investment level, we partially differentiate (7) with respect to $N$, which gives:

$$\frac{\partial}{\partial N} k^*_\Pi = -\frac{[A_2 - F A_1 - BN \cdot (F-1)] I}{(I \cdot (N+1) + 2F^2 Z_1 + 2Z_2)^2} - \frac{B \cdot (F-1)}{I \cdot (N+1) + 2F^2 Z_1 + 2Z_2}.$$

This partial derivative is strictly negative. Thus, it follows that the profit maximiser's storage-investment level decreases with the number of generating firms.

Next, to show the impact of the number of firms on the welfare maximiser's storage-investment level, we partially differentiate (12) with respect to $N$, giving:

$$\frac{\partial}{\partial N} k^*_W = -\frac{2[A_2 - F A_1 - BN \cdot (F-1)(N+2)] I \cdot (N+1)}{(I \cdot (N+1)^2 + F^2 Z_1 + Z_2)^2} - \frac{2B \cdot (F-1)(N+1)}{I \cdot (N+1)^2 + F^2 Z_1 + Z_2}.$$

This partial derivative is strictly negative, from which it follows that the welfare maximiser's storage-investment level decreases with the number of generating firms. ■

*Proof.* We have:

$$\begin{aligned} W(k^*_\Pi) - W(0) = &-\frac{1}{2} I B \cdot (F-1) N^3 \\ &- \left[2B \cdot (F-1)\left(F^2 Z_1 + Z_2\right) + I \cdot \left(\frac{1}{2}(A_2 - F A_1) + 2B(F-1)\right)\right] N^2 \\ &- \left[B \cdot (F-1)\left(\frac{3}{2} I + \frac{7}{2}\left(F^2 Z_1 + Z_2\right)\right)\right] N + \frac{1}{2}(A_2 - F A_1)\left[I + 3\left(F^2 Z_1 + Z_2\right)\right] = S(N). \end{aligned}$$

$S(N)$ is a cubic polynomial, which is strictly positive and decreasing at $N = 0$. $S(N)$, therefore, has exactly one positive root and either zero or two negative roots. Let $\tilde{N}$ denote the positive root of $S(N)$. $\tilde{N}$ is the critical number of firms, above which no storage yields higher social welfare than the profit maximiser's storage-investment level. ■

*Proof.* $W(k^*_\Pi) < W(0)$ means that an infinitesimal increase in the storage-investment level from $k^*_\Pi$ decreases social welfare. From Proposition 1, we know that this outcome is possible only when the number of firms is greater than $\bar{N}$. Hence, we must have that $\tilde{N} > \bar{N}$ when $W(k_\Pi^*) < W(0)$. ■

### 7.2 Market Equilibria with Linear Marginal Generation Costs

Here, we investigate how storage investment is affected by linear marginal generation costs. All modelling assumptions are the same as those that are in Sections 2–4, with the exception of the generation cost. We now assume that generation costs have the quadratic form:

$$c(g_{n,t}) = Bg_{n,t} + \frac{1}{2}NKg_{n,t}^2, n = 1, \ldots, N,$$

where $B, K > 0$.

We proceed with this analysis in four steps. We first derive the equilibrium production levels of the generating firms in the two operating periods. Next, we determine equilibrium storage-operation and -investment decisions for the profit- and welfare-maximising storage operators, respectively. Finally, we find the benchmark generation and storage-related decisions if all of them are made by a single social planner.

#### *7.2.1 Generator Equilibrium*

With linear marginal generation costs, generator $n$'s profit-maximisation problem becomes:

$$\max_{g_{n,1}, g_{n,2} \geq 0} P_1(g_1^G - Fd)g_{n,1} - Bg_{n,1} - \frac{1}{2}NKg_{n,1}^2 + P_2(g_2^G + d)g_{n,2} - Bg_{n,2} - \frac{1}{2}NKg_{n,2}^2.$$

Because this is a convex optimisation problem, its KKT conditions, which are:

$$0 \leq -A_1 + Z_1 \cdot (g_1^G - Fd) + Z_1 g_{n,1} + B + NKg_{n,1} \perp g_{n,1} \geq 0,$$

and:

$$0 \leq -A_2 + Z_2 \cdot (g_2^G + d) + Z_2 g_{n,2} + B + NKg_{n,2} \perp g_{n,2} \geq 0,$$

are sufficient for a global optimum. We can appeal to Lemma 1 to conclude that the equilibrium production levels of the generators are symmetric in each of the two periods. Adding the assumption that we have an interior solution (otherwise we have $g_t^G = 0$ in at least one period, which is an uninteresting case), gives:

$$g_1^G(d) = \frac{N \cdot (A_1 + Z_1 Fd - B)}{L_1}, \tag{23}$$

and:

$$g_2^G(d) = \frac{N \cdot (A_2 - Z_2 d - B)}{L_2}, \tag{24}$$

as the aggregate production levels of the generators in the two periods, where we define:

$$L_1 = Z_1 \cdot (N + 1) + NK,$$

and:

$$L_2 = Z_2 \cdot (N + 1) + NK.$$

We finally substitute these aggregate production levels into the period-1 and-2 inverse demand function to obtain equilibrium prices:

$$p_1(d) = \frac{(A_1 + Z_1 Fd)(Z_1 + NK) + Z_1 BN}{L_1}, \tag{25}$$

and:

$$p_2(d) = \frac{(A_2 - Z_2 d)(Z_2 + NK) + Z_2 BN}{L_2}, \tag{26}$$

in the two periods.

*7.2.2 Investment and Operating Equilibrium of Profit-Maximising Storage Operator*

We analyse the behaviour of a profit-maximising storage operator by first examining its operating decisions, which are governed by the profit-maximisation problem:

$$\begin{array}{lll} \max\limits_{d} & d \cdot [p_2(d) - F p_1(d)] & \\ \text{s.t.} & 0 \le d \le k, & (\mu) \end{array}$$

where $\mu$ is the Lagrange multiplier associated with the storage-capacity constraint and we use (25) and (26) as the prices in the two periods. Because $p_1(d)$ and $p_2(d)$ depend on $d$ linearly, this profit-maximisation is convex and its KKT conditions are necessary and sufficient for a global optimum. This storage operator's problem has the KKT conditions:

$$0 \le -(p_2(d) - F p_1(d)) - d p_2'(d) + d F p_1'(d) + \mu \perp d \ge 0,$$

$$0 \le k - d \perp \mu \ge 0,$$

which yields the same solution that is given in (5), with the *caveat* that we use the price functions that are given by (25) and (26), as opposed to (3) and (4).

Turning to the storage operator's investment decision, this is determined by the profit-maximisation problem:

$$\max_{k} [p_2(k) - F p_1(k)]k - (Ik^2)/2,$$

where by Lemma 2 we know that storage is fully utilised in the operating stage, meaning that $d = k$. We further assume that we have an interior solution, in which storage capacity is built (*i.e.*, that the non-negativity constraint, $k \ge 0$, is non-binding). The sufficient KKT condition for this problem is:

$$-p_2(k) + F p_1(k) + Ik - k p_2'(k) + F k p_1'(k) = 0.$$

Substituting (25) and (26), respectively, for the periods-1 and -2 price functions and solving gives:

$$k_\Pi^* = \frac{L_1 \cdot [A_2 \cdot (Z_2 + NK) + Z_2 BN] - F L_2 \cdot [A_1 \cdot (Z_1 + NK) + Z_1 BN]}{I L_1 L_2 + 2F^2 Z_1 L_2 \cdot (Z_1 + NK) + 2 Z_2 L_1 \cdot (Z_2 + NK)}, \tag{27}$$

as the profit maximiser's storage-investment level.

*7.2.3 Investment and Operating Equilibrium of Welfare-Maximising Storage Operator*

We analyse the case of a welfare-maximising storage operator in the same way that we do in the case of constant marginal generation costs. We begin by first deriving expressions for periods-1 and -2 consumer welfare, which are:

$$W_1^C(k) = \int_0^{g_1^G(k) - Fk} [P_1(x) - p_1(k)] dx = \frac{1}{2} Z_1 \cdot [g_1^G(k) - Fk]^2,$$

and:

$$W_2^C(k) = \int_0^{g_2^G(k) + k} [P_2(x) - p_2(k)] dx = \frac{1}{2} Z_2 \cdot [g_2^G(k) + k]^2,$$

respectively. These expressions are all written as functions of $k$, because we know from Lemma 2 that $d = k$ in a welfare-maximising equilibrium. Periods-1 and -2 producer welfare are, similarly:

$$W_1^G(k) = g_1^G(k) \cdot \left[p_1(k) - B - \frac{1}{2}Kg_1^G(k)\right] = \frac{2Z_1 + NK}{2N} g_1^G(k)^2,$$

and:

$$W_2^G(k) = g_2^G(k) \cdot \left[p_2(k) - B - \frac{1}{2}Kg_2^G(k)\right] = \frac{2Z_2 + NK}{2N} g_2^G(k)^2,$$

respectively. The welfare of the storage operator is given by:

$$W^S(k) = (2k_\Pi^* - k)k \frac{IL_1L_2/2 + F^2 Z_1 L_2 \cdot (Z_1 + NK) + Z_2 L_1 (Z_2 + NK)}{L_1 L_2},$$

where $k_\Pi^*$ is the value that is given by (27). Thus, substituting (23) and (24) for $g_1^G(d)$ and $g_2^G(d)$, we have:

$$\begin{aligned} W'(k) = FZ_1 & \frac{L_2^2 \cdot \left\{Z_1 N \cdot (A_1 - B) + F \cdot \left[Z_1^2(2N+1) + Z_1 NK \cdot (N+2) + N^2K^2\right] k\right\}}{L_1^2 L_2^2} \\ & + Z_2 \frac{L_1^2 \cdot \left[-Z_2 N \cdot (A_2 - B) + \left[Z_2^2(2N+1) + Z_2 NK \cdot (N+2) + N^2K^2\right] k\right]}{L_1^2 L_2^2} \\ & + (k_\Pi^* - k) \left\{ I + 2 \frac{\left[F^2 Z_1 L_1 L_2^2 \cdot (Z_1 + NK) + Z_2 L_1^2 L_2 \cdot (Z_2 + NK)\right]}{L_1^2 L_2^2} \right\}. \end{aligned} \tag{28}$$

Assuming that we have an interior solution (*i.e.*, that $k > 0$), the investment problem can be written as:

$$\max_k \, [W_1^C(k) - W_1^C(0)] + [W_2^C(k) - W_2^C(0)] + [W_1^G(k) - W_1^G(0)] + [W_2^G(k) - W_2^G(0)] + W^S(k),$$

which is a convex quadratic program. Using (28), the sufficient KKT condition gives:

$$k_W^* = \frac{A_2 L_1^2 J_2 - FA_1 L_2^2 J_1 + BN \cdot \left[Z_2 L_1^2 \cdot (L_2 + Z_2) - FZ_1 L_2^2 \cdot (L_1 + Z_1)\right]}{IL_1^2 L_2^2 + F^2 Z_1 L_2^2 \cdot \left[Z_1^2 + Z_1 \kappa + N^2K^2\right] + Z_2 L_1^2 \cdot \left[Z_2^2 + Z_2 \kappa + N^2 K^2\right]},$$

as the welfare maximiser's storage-investment level, where:

$$J_1 = (Z_1 + NK)^2 + Z_1 N^2 K,$$

$$J_2 = (Z_2 + NK)^2 + Z_2 N^2 K,$$

and:

$$\kappa = NK \cdot (N + 2).$$

### *7.2.4 Benchmark Central Planner's Problem*

We know from Section 4, and (16) in particular, that building storage is suboptimal for a social planner with constant marginal generation costs. This may not be the case, however, with linear marginal generation costs. As a benchmark, we consider the following problem:

$$\max_{g_1, g_2, k \geq 0} A_1(g_1 - Fk) - \frac{Z_1}{2}(g_1 - Fk)^2 + A_2(g_2 + k) - \frac{Z_2}{2}(g_2 + k)^2 - B \cdot (g_1 + g_2) - \frac{K}{2}\left(g_1^2 + g_2^2\right) - \frac{1}{2}Ik^2,$$

in which a welfare-maximising central planner owns and operates all generation and storage facilities. This is a convex quadratic optimisation problem, thus its KKT conditions are sufficient for a global optimum. Assuming an interior solution (*i.e.*, that generation levels are non-zero in both periods and that some energy storage is built), the KKT conditions give the optimal solution:

$$\begin{aligned} g_1^P &= \frac{A_1 - B}{Z_1 + K} \\ &\quad + \frac{FZ_1}{Z_1 + K} \cdot \frac{K \cdot [A_2 \cdot (Z_1 + K) - FA_1 \cdot (Z_2 + K)] - B \cdot [Z_1 Z_2 \cdot (F - 1) - K \cdot (Z_2 - FZ_1)}{I \cdot (Z_1 + K)(Z_2 + K) + K \cdot [Z_2 \cdot (Z_1 + K) + F^2 Z_1 \cdot (Z_2 + K)]}, \\ g_2^P &= \frac{A_2 - B}{Z_2 + K} \\ &\quad - \frac{Z_2}{Z_2 + K} \cdot \frac{K \cdot [A_2 \cdot (Z_1 + K) - FA_1 \cdot (Z_2 + K)] - B \cdot [Z_1 Z_2 \cdot (F - 1) - K \cdot (Z_2 - FZ_1)]}{I \cdot (Z_1 + K)(Z_2 + K) + K \cdot [Z_2 \cdot (Z_1 + K) + F^2 Z_1 \cdot (Z_2 + K)]}, \end{aligned}$$

and:

$$k^P = \frac{K \cdot [A_2 \cdot (Z_1 + K) - FA_1 \cdot (Z_2 + K)] - B \cdot [Z_1 Z_2 \cdot (F - 1) - K \cdot (Z_2 - FZ_1)]}{I \cdot (Z_1 + K)(Z_2 + K) + K \cdot [Z_2 \cdot (Z_1 + K) + F^2 Z_1 \cdot (Z_2 + K)]}.$$

$k^P$ is positive so long as the linear portion of the marginal generation cost is relatively high, *i.e.*, if $K \cdot (g_2^P - Fg_1^P) > B \cdot (F - 1)$. Hence, due to the linear marginal generation cost, storage may be required even under central planning because it enables the substitution of relatively inexpensive generation that is stored off-peak to displace higher-cost generation in the on-peak period.