The Valley of Death for New Energy Technologies: Appendix

1 Numbered equations from the paper

First we have the equations specifying the model:

$$U = \int_0^\infty e^{-\beta\tau} \frac{c(\tau)^{1-\gamma}}{1-\gamma}\, d\tau \tag{1}$$

$$\dot{k} = i - \delta k \tag{2}$$

$$\dot{k}_B = i_B - \delta k_B \tag{3}$$

$$\dot{k}_R = i_R - \delta k_R \tag{4}$$

$$Fk = \rho_R G k_R + \rho_B H k_B \tag{5}$$

$$\dot{S} = \rho_R Q k_R \tag{6}$$

$$\dot{N} = n \tag{7}$$

$$\mu(S, N) = \alpha_0 + \frac{\alpha_1}{\bar{S} - S - \alpha_2/(\alpha_3 + N)} = \alpha_0 + \frac{\alpha_1(\alpha_3 + N)}{(\bar{S} - S)(\alpha_3 + N) - \alpha_2} \tag{8}$$

$$\dot{H} = \begin{cases} b k_B^{\psi} j^{\alpha-\psi} & \text{if } H \leq \bar{H}, \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

$$Ak = c + i + i_R + n + i_B + j + \mu(S, N)\rho_R k_R + m\rho_B k_B \tag{10}$$

This leads to the objective for the constrained optimization problem:

$$\begin{aligned}\mathcal{H} = &\frac{c^{1-\gamma}}{1-\gamma} + q(i-\delta k) + q_R(i_R - \delta k_R) + q_B(i_B - \delta k_B) + \nu n + \sigma\rho_R Q k_R \\ &+ \eta b k_B^{\psi} j^{\alpha-\psi} + \lambda\Big\{Ak - c - i - i_R - i_B - n - j - \mu(S,N)\rho_R k_R - m\rho_B k_B\Big\} \\ &+ p_e\big\{\rho_R G k_R + \rho_B H k_B - Fk\big\} + \theta_{RL}\rho_R + \theta_{RU}(1-\rho_R) + \theta_{BL}\rho_B \\ &+ \theta_{BU}(1-\rho_B) + \omega i + \omega_R i_R + \omega_B i_B + \omega_N n + \omega_H j\end{aligned} \tag{11}$$

The first order conditions for a maximum of (11) with respect to the control variables are:

$$\frac{\partial \mathcal{H}}{\partial c} = c^{-\gamma} - \lambda = 0 \tag{12}$$

$$\begin{aligned}&\frac{\partial \mathcal{H}}{\partial \rho_R} = \sigma Q k_R - \lambda k_R \mu(S,N) + p_e G k_R + \theta_{RL} - \theta_{RU} = 0 \\ &\theta_{RL}\rho_R = 0, \theta_{RL} \geq 0, \rho_R \geq 0, \theta_{RU}(1-\rho_R) = 0, \theta_{RU} \geq 0, \rho_R \leq 1\end{aligned} \tag{13}$$

$$\begin{aligned}&\frac{\partial \mathcal{H}}{\partial \rho_B} = -\lambda m k_B + p_e H k_B + \theta_{BL} - \theta_{BU} = 0 \\ &\theta_{BL}\rho_B = 0, \theta_{BL} \geq 0, \rho_B \geq 0, \theta_{BU}(1-\rho_B) = 0, \theta_{BU} \geq 0, \rho_B \leq 1\end{aligned} \tag{14}$$

$$\frac{\partial \mathcal{H}}{\partial i} = q - \lambda + \omega = 0; \omega i = 0, \omega \geq 0, i \geq 0 \tag{15}$$

$$\frac{\partial \mathcal{H}}{\partial i_R} = q_R - \lambda + \omega_R = 0; \omega_R i_R = 0, \omega_R \geq 0, i_R \geq 0 \tag{16}$$

$$\frac{\partial \mathcal{H}}{\partial i_B} = q_B - \lambda + \omega_B = 0; \omega_B i_B = 0, \omega_B \geq 0, i_B \geq 0 \tag{17}$$

$$\frac{\partial \mathcal{H}}{\partial n} = \nu - \lambda + \omega_N = 0, \omega_N n = 0, \omega_N \geq 0, n \geq 0 \tag{18}$$

$$\frac{\partial \mathcal{H}}{\partial j} = \eta(\alpha-\psi) b k_B^{\psi} j^{\alpha-\psi-1} - \lambda + \omega_H = 0, \omega_H j = 0, \omega_H \geq 0, j \geq 0 \tag{19}$$

The differential equations for the co-state variables are:

$$\dot{q} = \beta q - \frac{\partial \mathcal{H}}{\partial k} = (\beta+\delta)q - \lambda A + p_e F \tag{20}$$

$$\dot{q}_R = \beta q_R - \frac{\partial \mathcal{H}}{\partial k_R} = (\beta+\delta)q_R - \sigma\rho_R Q + \rho_R\lambda\mu(S,N) - \rho_R p_e G \tag{21}$$

$$\dot{q}_B = \beta q_B - \frac{\partial \mathcal{H}}{\partial k_B} = (\beta + \delta) q_B - \eta \psi b k_B^{\psi - 1} j^{\alpha - \psi} + \rho_B \lambda m - \rho_B p_e H \tag{22}$$

$$\dot{\nu} = \beta \nu - \frac{\partial \mathcal{H}}{\partial N} = \beta \nu + \lambda \rho_R k_R \frac{\partial \mu}{\partial N} \tag{23}$$

$$\dot{\sigma} = \beta \sigma - \frac{\partial \mathcal{H}}{\partial S} = \beta \sigma + \lambda \rho_R k_R \frac{\partial \mu}{\partial S} \tag{24}$$

$$\dot{\eta} = \beta \eta - \frac{\partial \mathcal{H}}{\partial H} = \beta \eta - \rho_B p_e k_B \tag{25}$$

We then defined various critical energy prices:

$$p_{LR}^R = \frac{\lambda (A + \mu) - \sigma Q}{F + G} \tag{26}$$

$$p_{LR}^B = \frac{\lambda}{F + H} \left[A + m - \frac{\psi}{\alpha - \psi} k_B^{(\alpha - 1)s} Y \right] \tag{27}$$

$$p_{\rho 0}^B = \frac{\lambda}{F} \left[A - \frac{\psi}{\alpha - \psi} k_B^{(\alpha - 1)s} Y \right] \tag{28}$$

where we have defined $Y \equiv [\eta (\alpha - \psi) b / \lambda]^s$ and $s \equiv 1/(1 + \psi - \alpha) > 1$

$$p_{SR}^R = \frac{\lambda \mu - \sigma Q}{G} \tag{29}$$

$$p_{SR}^B = \frac{\lambda m}{H} \tag{30}$$

The energy market equilibrium condition relating the use of k_B to the use of k_R

$$\rho_B = \frac{F k - \rho_R G k_R}{H k_B} \tag{31}$$

Setting the fossil fuel production costs in the calibration:

$$0.1614 = \alpha_0 + \frac{\alpha_1}{\bar{S} - \alpha_2 / \alpha_3} \tag{32}$$

2 Differential equations in each regime going backwards in time

2.1 The long run endogenous growth economy

Beyond T_H, H is constant at $\bar{H}$. The control variables are c, i and i_B, while the state variables are k and k_B. In this regime, the resource constraint (10) simplifies to

$$c + i + i_B + mk_B = Ak \tag{33}$$

while the energy market equilibrium condition becomes

$$Fk = \bar{H}k_B \tag{34}$$

Differentiating (34) and using the assumption that the depreciation rates are identical, we obtain

$$Fi = \bar{H}i_B \tag{35}$$

With both $i, i_B > 0$, (15) and (17) imply $q = \lambda = q_B$. Noting also that $j = 0$ and $\rho_B = 1$, the co-state equations for q and q_B in this regime then imply

$$\dot{\lambda} = (\beta + \delta)\lambda - \lambda A + p_e F = (\beta + \delta)\lambda + \lambda m - p_e \bar{H} \tag{36}$$

In particular, the price of energy is constant at

$$p_e = \frac{A + m}{\bar{H} + F}\lambda \tag{37}$$

while λ satisfies the differential equation

$$\frac{\dot{\lambda}}{\lambda} = \beta + \delta - \frac{A\bar{H} - mF}{\bar{H} + F} \equiv -\bar{A} \tag{38}$$

where $\bar{A}$ is a constant.[1] For another constant $\bar{K}$, the solution to (38) can be written

$$\lambda = \bar{K}e^{-\bar{A}t} \tag{39}$$

Using the differential equation for k, (39) and the first order condition (12) for c, resource constraint (33), the constraint on investment (35) and the definition of $\bar{A}$ in (38) we get

$$\dot{k} = (\bar{A} + \beta)k - \frac{\bar{H}\bar{K}^{-1/\gamma}}{\bar{H} + F}e^{\bar{A}t/\gamma} \tag{40}$$

The integrating factor for the differential equation (40) is $e^{-(\bar{A}+\beta)t}$, so the solution can be written

$$k = C_0 e^{(\bar{A}+\beta)t} + \frac{\bar{H}\gamma\bar{K}^{-1/\gamma}}{(\bar{H} + F)[\beta\gamma + \bar{A}(\gamma - 1)]}e^{\bar{A}t/\gamma} \tag{41}$$

for another constant C_0. However, the transversality condition requires

$$\lim_{t\to\infty} e^{-\beta t}\lambda k = C_0\bar{K} + \lim_{t\to\infty} \frac{\bar{H}\gamma\bar{K}^{(1-1/\gamma)}}{(\bar{H} + F)[\beta\gamma + \bar{A}(\gamma - 1)]}e^{(\bar{A}/\gamma - \bar{A} - \beta)t} = 0 \tag{42}$$

that is, $C_0 = 0$ and[2] $\bar{A}(1 - \gamma) < \beta\gamma$. Thus, k will have a growth rate $\bar{A}/\gamma$ and be given by

$$k = \frac{\bar{H}\gamma\bar{K}^{-1/\gamma}e^{\bar{A}t/\gamma}}{(\bar{H} + F)[\beta\gamma + \bar{A}(\gamma - 1)]} \tag{43}$$

with $\lambda = q = q_B$ given by (39) and where $\bar{K}$ is a constant yet to be determined. From (34) and (43), the capital stock allocated to renewable energy production will be

$$k_B = \frac{F\gamma\bar{K}^{-1/\gamma}e^{\bar{A}t/\gamma}}{(\bar{H} + F)[\beta\gamma + \bar{A}(\gamma - 1)]} \tag{44}$$

The beginning of the final regime occurs at T_H when H attains $\bar{H}$ and $\eta = 0$. The value of k, k_B, λ and φ at T_H set their values at the end of regime 5.

[1] To get perpetual growth, we must have $c \to \infty$ as $t \to \infty$, which from (12) will require $\lambda \to 0$ and hence $\bar{A} > 0$, that is $A > \beta + \delta + F(\beta + \delta + m)/\bar{H}$

[2] Since $\bar{A} > 0$, the inequality will be satisfied if $\gamma > 1$, as assumed in the numerical analysis. If $0 < \gamma < 1$, it requires an additional restriction, $A < \beta/(1 - \gamma) + \delta + F[\beta/(1 - \gamma) + \delta + m]/\bar{H}$, on the range of parameter values.

2.2 Regime 5: Fully dynamic renewable regime

Regime 5 has direct investment in renewable energy R&D ($j > 0$), end-use capital ($i > 0$) and renewable energy production capital ($i_B > 0$). Using the solutions for c from (12) and j from (19), and $\rho_B = 1$, the resource constraint (10) gives one equation linking i and i_B

$$i + i_B = Ak - Yk_B^{s\psi} - mk_B - \lambda^{-1/\gamma} \tag{45}$$

Differentiating the energy market equilibrium condition $Fk = Hk_B$ and using this condition with (2), (3) and (19), which implies $\dot{H} = bk_B^{s\psi}Y^{\alpha-\psi}$, we obtain a second equation linking i and i_B

$$Fi - Hi_B = bk_B^{s\psi+1}Y^{\alpha-\psi} \tag{46}$$

Equations (46) and (45) then give us two equations to solve for i and i_B, as illustrated in Figure 1. The differential equations (2) and (3) then yield $\dot{k}$ and $\dot{k}_B$. In this regime, we will again have

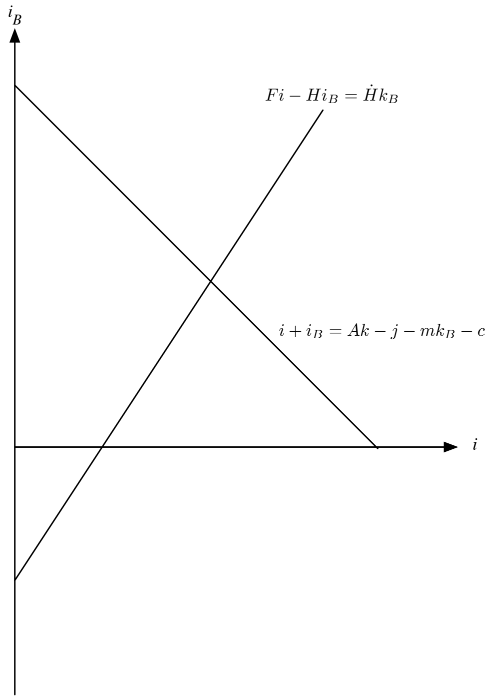

Figure 1: Solving for investments in regime 5

$q = \lambda = q_B$. Noting that the price of energy p_e will be given by (27) and $\rho_B = 1$, the co-state differential equations are $\dot{\eta} = \beta\eta - p_e k_B$ and $\dot{\lambda} = (\beta + \delta - A)\lambda + p_e F$.

2.3 Regime 4: No investment in renewable capacity

The incentive to invest in H will decline as $H \uparrow \bar{H}$. Solving *backwards* in time in regime 5, we therefore expect $\dot{H}/H$ to increase, shifting the upward sloping line in Figure 1 to the right. On the other hand, as we move backwards in time, the resources available to support $i + i_B$ will decrease, shifting the downward sloping line to the left. Thus, i_B is likely to decline rapidly and, as shown in the text, the constraint $i_B \geq 0$ will bind at some $T_B > T_R$. We then enter regime 4, which involves full use of renewable capacity ($\rho_B = 1$) but no investment in additional capacity (so $\dot{k}_B = -\delta k_B$). The lower boundary of regime 4 is T_R where energy production from fossil fuels ends.

Using the solutions for c from (12) and j from (19), the resource constraint (10) implies:

$$i = Ak - Yk_B^{s\psi} - mk_B - \lambda^{-1/\gamma} \tag{47}$$

However, as in regime 5, the energy market equilibrium condition will also determine a value for i:

$$Fi = bk_B^{s\psi+1}Y^{\alpha-\psi} \tag{48}$$

Equating the two expressions for i from (47) and (48) we obtain

$$bk_B^{s\psi+1}Y^{\alpha-\psi} + FYk_B^{s\psi} - FAk + mFk_B + \lambda^{-1/\gamma}F = 0 \tag{49}$$

In preparation for differentiating (49) note that, since $i > 0$, (15) implies $q = \lambda$ and (20) allows us to write the derivative of λ in terms of p_e as

$$\dot{\lambda} = (\beta + \delta - A)\lambda + p_e F \tag{50}$$

Now use (50) and (25) to obtain the derivative of η/λ:

$$\frac{d}{dt}\left(\frac{\eta}{\lambda}\right) = -\frac{p_e}{\lambda}\left[k_B + F\frac{\eta}{\lambda}\right] + (A - \delta)\frac{\eta}{\lambda} \tag{51}$$

Also, using the definition of Y, we obtain:

$$\dot{Y} = (s-1)bY^{\alpha-\psi}\frac{d}{dt}(\frac{\eta}{\lambda}) \tag{52}$$

The derivative of (49) can then be written in terms of these expressions as:

$$k_B^{s\psi}\left[F\dot{Y} - s\psi\delta FY - b\delta(s\psi+1)k_BY^{\alpha-\psi} + k_B\frac{\lambda}{\eta}\dot{Y}\right] + F\left[A\delta k - Ai - m\delta k_B - \frac{1}{\gamma}\lambda^{-\frac{1+\gamma}{\gamma}}\dot{\lambda}\right] = 0 \tag{53}$$

which can be solved for an energy price p_e that will ensure (47) and (48) give the same solution for i. Co-state variable η will evolve according to (23) with $\rho_B = 1$.

At the lower boundary T_R of regime 4, ρ_R jumps from one to zero, while ρ_B jumps from zero to one. Also, p_e equals the two short-run costs of energy production, (30) and (29). Using the fact that σ converges to zero at T_R, we therefore must have

$$\frac{p_e}{\lambda} = \frac{\mu}{G} = \frac{m}{H} \tag{54}$$

Equation (54) can be used to determine T_R and the value of S at T_R once $N(T_R) = \bar{N}$ has been specified. Also, given that k_B and H are known at T_R when solving backwards in time, energy market equilibrium will determine a limiting value for k_R at T_R, namely $k_R(T_R) = H(T_R)k_B(T_R)/G$.

2.4 Regime 3: Only fossil fuels used, $i_R = n = i = 0$

In this regime, only fossil fuels are used to produce energy ($\rho_R = 1, \rho_B = 0$). However, we have $i_R = 0$, so k_R declines according to $\dot{k}_R = -\delta k_R$. The energy market equilibrium condition now becomes $Fk = Gk_R$. However, since $i_R = 0$ and F and G are constant, differentiation implies we now must also have $i = 0$. We also have $n = 0$, so N remains fixed at $\bar{N}$. Using the solutions for c and j, the resource constraint (10) can be written

$$i_B = Ak - Yk_B^{s\psi} - \mu k_R - \lambda^{-1/\gamma} \tag{55}$$

From $i_B > 0$ and (17) we get $q_B = \lambda$ and $\dot{q}_B = \dot{\lambda}$. Also, $\rho_B = 0$ in regime 3, as it does for all

$t < T_R$. Then from (22), $\dot{\lambda}$ will evolve in regime 3 according to

$$\frac{\dot{\lambda}}{\lambda} = \beta + \delta - \frac{\psi}{\alpha - \psi} k_B^{(\alpha-1)s} Y \tag{56}$$

The co-state variables ν, σ and η evolve according to (21), (23), (24) and (25) with $\rho_R = 1$ and $\rho_B = 0$ and $p_e = p_{SR}^R$. In particular, for all $t < T_R$, η will evolve according to[3]

$$\dot{\eta} = \beta\eta \tag{57}$$

Similarly, (24) with $\rho_R = 1$ implies

$$\dot{\sigma} = \beta\sigma + \lambda k_R \frac{\partial \mu}{\partial S} \tag{58}$$

while (23) with $\rho_R = 1$ implies

$$\dot{\nu} = \beta\nu + \lambda k_R \frac{\partial \mu}{\partial N} \tag{59}$$

The lower boundary T_N of regime 3 will be where $\nu = \lambda$.

2.5 Regime 2: Only fossil fuels used, investment in N but not k_R

For $T \in [T_Q, T_N]$, again $\rho_R = 1$ and $\rho_B = 0$, while $i_R = 0$ again implies k_R declines according to $\dot{k}_R = -\delta k_R$. As in regime 3, energy market equilibrium will imply that $i = 0$ and $\dot{k} = -\delta k$, but $i_B, j, n > 0$. Using the solutions for c and j, and $i = 0 = i_R$, the resource constraint (10) implies

$$i_B + n = Ak - Y k_B^{s\psi} - \mu k_R - \lambda^{-1/\gamma} \tag{60}$$

Since $n > 0$ for all $t \leq T_N$, $\nu = \lambda$ and hence $\dot{\nu} = \dot{\lambda}$. From $i_B > 0$ and (17), $\lambda = q_B$, and using $\rho_B = 0$ and (22), λ again evolves according to (56). Then, using also (23) and $\rho_R = 1$, we obtain:

$$k_R \frac{\partial \mu}{\partial N} = \delta - \frac{\psi}{\alpha - \psi} k_B^{(\alpha-1)s} Y \tag{61}$$

[3] Thus, $\eta > 0$ and increasing exponentially for $t < T_R$, while for $t > T_R$ it decreases to zero at T_H.

Then (noting that $(\alpha-1)s = \psi s - 1$ and $i_R = 0$) the derivative of (61) can be written as:

$$
\begin{aligned}
&-\delta\frac{\partial\mu}{\partial N}k_R + k_R\frac{\partial^2\mu}{\partial N^2}n + Qk_R^2\frac{\partial^2\mu}{\partial S\partial N} + \\
&\frac{\psi s k_B^{\psi s-2}}{\alpha-\psi}\left[(\alpha-1)Y(i_B-\delta k_B) + (\alpha-\psi)bk_BY^{\alpha-\psi}\frac{d}{dt}(\frac{\eta}{\lambda})\right] = 0
\end{aligned}
\tag{62}
$$

where (25), $\rho_B = 0$, (56) and (61) yield the derivative of η/λ in this regime:

$$\frac{d}{dt}(\frac{\eta}{\lambda}) = -\frac{\eta}{\lambda}\frac{\partial\mu}{\partial N}k_R \tag{63}$$

The two equations (60) and (62) can then be solved for the two investments i_B and n. Using $\nu = \lambda$ we find that $\dot{\lambda}/\lambda$ will now satisfy a much simpler equation

$$\frac{\dot{\lambda}}{\lambda} = \beta + k_R\frac{\partial\mu}{\partial N} \tag{64}$$

Throughout regimes 2 and 3, $p_e = p_{SR}^R$ and hence $\dot{q}_R = (\beta+\delta)q_R$. Then since $q_R = 0$ at T_R, q_R remains zero throughout regimes 2 and 3. However, at the lower boundary of regime 2, we must have $q_R = \lambda > 0$ since $i_R > 0$ throughout regime 1 and falls to zero only at T_Q. The co-state variable q_R must therefore be left continuous and differentiable for all $t < T_Q$, but jump discontinuously to zero at T_Q when investment in k_R ceases. The lower boundary T_Q of regime 2 cannot be calculated endogenously and becomes an additional value that has to be set to attain the initial conditions.

The differential equations for the co-state variables η and σ will be (57) and (58) as in regime 3. However, σ also is discontinuous at T_Q. Throughout regime 1 and also as $t \uparrow T_Q$, $p_e = p_{LR}^R = p_{\rho 0}^B$, and to ensure this relationship in regime 1 when solving backwards we set:

$$\sigma(T_Q) = \frac{\lambda}{FQ}\left[\mu F + \frac{\psi(F+G)}{\alpha-\psi}k_B^{(\alpha-1)s}Y - AG\right] \tag{65}$$

2.6 Regime 1: Investment in both k_R and k_B but only fossil fuel is used

In regime 1, renewable energy is not produced ($\rho_B = 0$), but all energy investments i_R, i_B, n and j are positive. Using the solutions for c and j, the resource constraint (10) can now be written

$$i + i_B + i_R + n = Ak - \mu k_R - Yk_B^{s\psi} - \lambda^{-1/\gamma} \tag{66}$$

Once again, the energy market equilibrium condition can be differentiated to yield

$$Fi - Gi_R = 0 \tag{67}$$

and hence $i = Gi_R/F$. A third equation involving the investments can again be obtained from (61). Since $i_R > 0$, however, (62) is modified to an equation involving i_R, i_B and n:

$$\begin{aligned} &\frac{\partial\mu}{\partial N}i_R - \delta\frac{\partial\mu}{\partial N}k_R + k_R\frac{\partial^2\mu}{\partial N^2}n + Qk_R^2\frac{\partial^2\mu}{\partial S\partial N} + \\ &\frac{\psi s k_B^{\psi s-2}}{\alpha-\psi}\left[(\alpha-1)Y(i_B-\delta k_B) + (\alpha-\psi)bk_BY^{\alpha-\psi}\frac{d}{dt}(\frac{\eta}{\lambda})\right] = 0 \end{aligned} \tag{68}$$

where the derivative of η/λ is once again given by (63). The fourth equation involving investments arises from the fact that, with both $i_R, i_B > 0$, p_e has to equal both (26) and (28):

$$\mu - \frac{\sigma}{\lambda}Q - \frac{AG}{F} + \frac{\psi k_B^{(\alpha-1)s}Y(F+G)}{(\alpha-\psi)F} = 0 \tag{69}$$

Then differentiating (69) we obtain:

$$\begin{aligned} &\frac{\partial\mu}{\partial N}n + \frac{\sigma Q}{\lambda}\left[\frac{\partial\mu}{\partial N}k_R - \pi\right] + \\ &\frac{\psi s(F+G)k_B^{\psi s-2}}{(\alpha-\psi)F}\left[(\alpha-1)Y(i_B-\delta k_B) + (\alpha-\psi)bk_BY^{\alpha-\psi}\frac{d}{dt}(\frac{\eta}{\lambda})\right] = 0 \end{aligned} \tag{70}$$

where we have used $\nu = \lambda$ and thus (23) and (24) with $\rho_R = 1$ to obtain:

$$\frac{d}{dt}(\frac{\sigma}{\lambda}) = \left[\frac{\partial\mu}{\partial S} - \frac{\sigma}{\lambda}\frac{\partial\mu}{\partial N}\right]k_R \tag{71}$$

The four equations (66), (67), (68) and (70) can then be solved for i, i_R, i_B and n. The differential equations governing the evolution of the co-state variables will again have $\rho_R = 1$, $\rho_B = 0$ and p_e given by (28). In particular, $\dot{\lambda}/\lambda$ (with $q_R = q_B = \lambda$) will satisfy the simpler equation (64), while $\dot{\eta}$ and $\dot{\sigma}$ will again satisfy (57) and (58) respectively.

3 Initial and terminal conditions

At $t = 0$, there are three initial conditions for the physical capital stocks $k(0)$, $k_R(0)$ and $k_B(0)$, an initial value for renewable energy productivity $H(0)$ and, by definition, $S(0) = N(0) = 0$. However, active investment in N, k_R and k_B imposes two constraints on these state variables, leaving only four independent targets. We take these to be $k(0), H(0)$ and $S(0) = N(0) = 0$. We need to choose four initial values for the differential equations and solve backwards to hit these targets. The solution in the final analytical regime depends on an unknown constant $\bar{K}$, while we also need to specify values for T_H, N at T_R, and the time T_Q when investment in k_R ceases.

4 MatLab programs

The MatLab programs that were written to solve the system of differential equations for given parameter values are available as separate .m files on the same web site as this appendix.